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Finite Element Modeling of One-Dimensional Viscoelastic Structures Using Anelastic Displacement Fields

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A physically motivated approach to modeling the dynamic behavior of viscoelastic structures using augmenting thermodynamic fields was previously reported. Anelastic displacement fields, special kinds of augmenting thermodynamic fields, are now introduced. Instead of addressing physical damping mechanisms directly, as in the earlier approach, their effects on the displacement field are considered. In this approach, the total displacement field comprises two parts: 1) an elastic part and 2) an anelastic part. The material constitutive equations are developed, as well as the governing differential equations and boundary conditions for a one-dimensional structural member, and are compared to the results developed previously using the augmented thermodynamic fields approach. In addition, a physical interpretation of some of the quantities involved is advanced in terms of a classical mechanical analogy. Because the anelastic displacement field(s) and the total displacement field may be treated similarly in analytical or numerical study, the key practical benefit of this anelastic displacement fields approach is that it leads to the straightforward development of time-domain finite element models. Modal analyses and frequency response analyses have been implemented using the matrix manipulation capabilities of a commercial finite element code.

Introduction

VIBRATION damping is essential to the attainment of performance goals for engineered systems that exhibit significant structural dynamic response. Passive structural damping can be increased most predictably through the use of materials with known damping properties. Because of the potential for practical payoffs, some research efforts have pursued the development of structural materials (typically composites) with increased damping properties. The most common method used today to increase structural damping, however, involves the use of nonstructural materials, typically high-damping viscoelastic polymers.[1]

The mechanical properties of these damping materials are often sensitive to frequency, temperature, type of deformation (i.e., shear or dilation), and sometimes amplitude. To ensure design adequacy, performance is usually analytically evaluated at a few specific temperatures that span the expected operating range of the system of interest. Material properties appropriate to each temperature of interest are used in these analyses.

Although damping models currently available in commercial finite element software (e.g., viscous damping, proportional damping, hysteretic or structural damping, and viscous modal damping) do provide energy dissipation, in general they are not physically motivated. None of these models directly preserves the frequency-dependent behavior characteristic of real materials, and each suffers in practice from one difficulty or another. Although better accuracy is potentially available through the use of material models such as general viscoelasticity, such models are not widely used in engineering applications.

The most common approach to analysis of damping designs using viscoelastic polymers is perhaps the modal strain energy (MSE) method.[2] In this approach, a modal analysis of an elastic structure is performed using material properties appropriate to a specific temperature and frequency. Effective modal damping ratios are found for each normal vibration mode as a weighted sum of the damping of the constituent materials, where the weighting factor is the fraction of total modal strain energy stored in each. When several modal damping ratios are to be determined, use of the MSE method requires iteration. Because of the frequency dependence of material properties, suitable definition of each mode requires the use of properties appropriate to a frequency range containing that modal frequency.

Dynamic models based on the use of the MSE method and modal damping can have several drawbacks: iteration is required to determine the modal damping for each vibration mode in the frequency range of interest, the resulting modes are not orthogonal in the usual sense because of the lack of a single stiffness matrix, the relative phase of vibration at various points on a structure is neglected, and modes that are closely spaced in frequency may be predicted poorly. Such issues of model fidelity are especially important in structural control applications.[3]

Dissatisfaction with available damping modeling techniques has motivated considerable research on the subject of time-domain methods that capture the essential frequency dependence of viscoelastic material properties and that are compatible with current structural finite element analysis techniques. This research can be broadly classified into 1) those that use fractional time derivatives to model material relaxation behavior and 2) those that use integer time derivatives. Both approaches have advantages and disadvantages, and no approach is clearly superior to others in all cases.

In a series of papers, Bagley and Torvik[4,5] and Bagley and Calico[6] have developed a fractional derivative model of viscoelastic material behavior and applied it to a number of structural modeling and response problems. An important feature of their approach is the ability to capture the relatively weak frequency dependence exhibited by many materials using just a few, typically four, model parameters. This feature makes the fractional derivative model especially useful in frequency-domain analyses. Bagley and Torvik[5] initially developed frequency-domain finite elements that could be used to obtain structural responses for load histories that have Laplace transforms. In later work,[6] a time-domain version with fractional state equations was developed.

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In the time domain, the presence of fractional operators makes the solution of structural equations considerably more complicated than it is for those involving ordinary differential operators. Padovan[7] developed transient solution algorithms for finite element simulation of viscoelastic problems involving fractional operators. Enelund and Olsson[8] proposed using a different form of the time-domain equations of motion (one which simplifies the specification of initial conditions), as well as a different method for time discretization. Both of these solution approaches require that a truncated time history be retained for use, in effect adding coordinates to the structural dynamic model.

Structural equations with time-domain damping models involving ordinary integer differential operators are more straightforward to solve than those involving fractional derivative operators. Several of these kinds of models have been proposed in the literature, including the anelastic displacement fields (ADF) model,[9–11] the Golla–Hughes–McTavish (GHM) model,[12–14] and Yiu's model.[15,16]

The augmenting thermodynamic fields (ATF) method, the precursor to ADF, is a time-domain continuum model of material damping that preserves the characteristic frequency-dependent damping and modulus of real materials, that is, a physically motivated model compatible with current finite element structural analysis methods. In its initial development, the ATF method introduced a single augmenting field to model the behavior of materials and structures with light damping.[9] In subsequent work, using multiple ATFs, the ability to model high damping materials having relatively weak frequency dependence was developed.[10] However, this early work was effectively limited to structural members under uniaxial stress states, such as bars and beams. Dovstam[17] later based the development of a three-dimensional frequency-domain damping model on the ATF method, using anelastic strains.

In the present work, ADF, which are special kinds of ATF, are introduced. Instead of addressing physical damping mechanisms directly, as in the earlier approach, their effects on the displacement field are considered. The ADF method has several advantages over the initial ATF method; in particular, the ADF formulation has been generalized to three-dimensional problems,[11] to problems involving temperature dependence,[18] and to problems involving nonlinear strain dependence.[19–21]

The GHM model[12–14] and Yiu's model[15,16] are similar in some ways to the present ADF model. All of these models result in time-domain viscoelastic finite elements, using additional coordinates to more accurately model material behavior. The ADF method differs in that it involves a direct time-domain formulation, not transform based, and yields finite elements using conventional methods. The dissipation coordinates of the GHM method are internal to individual elements, whereas the anelastic displacement fields of ADF are continuous from element to element, reflecting its basis as a field theory. The internal unobservable degrees of freedom of Yiu's model are introduced as nodal variables using an analogy with a generalized lumped-parameter Maxwell model.

Because it was developed explicitly with second-order dynamics, the GHM method[12–14] is quite compatible with current structural analysis methods and has proven to be useful in practice. Both the ADF and Yiu's models[15,16] may also be readily expressed in second-order form. Although the second-order form of a GHM (minioscillator) model[12–14] can permit unrealistic material behavior, such as damping that decreases with the square of the frequency or a material modulus that decreases with increasing frequency, the latter difficulty at least can be avoided by proper selection of material model parameters. In its current state of development, Yiu's model[15,16] assumes a single loss factor for all material moduli (e.g., shear and bulk for an isotropic material). In many respects, however, finite element models that result from the use of the GHM method[12–14] and, especially, Yiu's method are quite similar to ADF models.

All three of these approaches have advantages over the conventional MSE method in that they yield linear time-domain finite element models, the frequency-dependent elastic and dissipative aspects of structural behavior are represented in fixed (not frequency-dependent) system matrices, modal damping is calculated concurrently with modal frequency without iteration, the resulting complex modes more accurately reflect the relative phase of vibration at various points, and modal orthogonality is preserved.

The purposes of this paper are to introduce and develop the ADF method for a one-dimensional structural member and to compare the results to those obtained previously using the ATF approach.

Theory for a One-Dimensional Member

This section introduces the notion of an ADF, along with the motivation for its use. In addition, it outlines the derivation of the material constitutive relations, as well as the governing differential equations and boundary conditions for a one-dimensional structural member, and compares the results to those previously obtained using the ATF approach. Finally, a physical interpretation of some of the quantities involved is advanced in terms of a classical mechanical analogy.

Anelastic Displacement Field

The initial development of the ATF damping modeling method was based on the notion of scalar internal variables or ATF that described the interaction of the displacement field with irreversible processes occurring at the materials level. Although this use of ATF was straightforward enough for the one-dimensional development of the method, the means to extend it to the general case of a three-dimensional continuum was not apparent.

Analogies with classical coupled-field problems having dissipative properties provided little guidance. For example, although thermoelastic damping in the three-dimensional case can be modeled using a single augmenting scalar field, namely, the temperature, that physical process is effective only in damping vibrations that involve volumetric deformation and, therefore, is inadequate as a general model of viscoelastic behavior. (Shear deformation is often assumed to be the only significant contributor to loss in high damping polymers.) Although electroelastic (or equivalently, piezoelectric) damping in the three-dimensional case can be modeled using a single augmenting field, namely, the electrical potential, not all types of deformation produce electrical fields and damping. An acceptable method must be capable of damping any type of cyclic deformation.

When a formulation for the three-dimensional problem based on the use of scalar ATF was not apparent, a different point of view was considered. Instead of addressing the physical mechanism itself, its effects on the displacement field were considered. Then, the total displacement field might be considered the sum of two parts: 1) an elastic part and 2) an anelastic part.[11] Although this would evidently require the use of an additional three-component vector field in continuum modeling, it might exhibit certain parallels with other parts of the displacement field, leading to potential simplifications in finite element modeling.

Consider the one-dimensional structural member shown in Fig. 1. Mass particles within the member are identified by their location in a reference configuration by the coordinate x. The total mechanical displacement of any point x at time t is given by $u(x, t)$.

Now, let this total displacement be expressed as the sum of elastic and anelastic displacements as follows:

$$u(x, t) = u^E(x, t) + u^A(x, t) \tag{1}$$

In this equation, $u^E(x, t)$ is the elastic displacement field and $u^A(x, t)$ is the ADF. The longitudinal normal strain (total strain) $\varepsilon(x, t)$ is given by

$$\varepsilon(x, t) = \frac{\partial u(x, t)}{\partial x} = \frac{\partial u^E(x, t)}{\partial x} + \frac{\partial u^A(x, t)}{\partial x}$$

$$= \varepsilon^E(x, t) + \varepsilon^A(x, t) \tag{2}$$

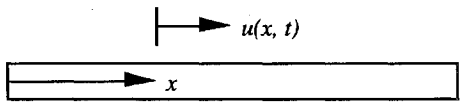

Fig. 1 One-dimensional problem geometry.

where $\varepsilon^E(x,t)$ is the elastic strain and $\varepsilon^A(x,t)$ is the anelastic (or creep) strain. Note that the strains associated with the elastic part of the displacement are those that are instantaneously proportional to the stress.

Material Constitutive Equation

The constitutive equation for a linear elastic material in uniaxial stress is given by

$$\sigma = E_u \varepsilon^E \tag{3a}$$

where E_u is the unrelaxed modulus (or high-frequency, dynamic, adiabatic) modulus. This relation can be expressed in terms of the total strain and the anelastic strain as follows:

$$\sigma = E_u(\varepsilon - \varepsilon^A) \tag{3b}$$

An additional constitutive equation may be found from thermodynamic considerations. Such an equation would express the dependence of σ^A, the anelastic stress (a quantity thermodynamically conjugate to ε^A), on ε and ε^A. To develop this relation, consider the following trial quadratic Helmholtz free energy density function f:

$$f = \frac{1}{2}E_u\left(\varepsilon^2 - 2\varepsilon\varepsilon^A + c{\varepsilon^A}^2\right) = \frac{1}{2}E_u\begin{Bmatrix}\varepsilon\\ \varepsilon^A\end{Bmatrix}^T\begin{bmatrix}1 & -1\\ -1 & c\end{bmatrix}\begin{Bmatrix}\varepsilon\\ \varepsilon^A\end{Bmatrix} \tag{4}$$

This yields the following constitutive equation for the stress:

$$\sigma = \frac{\partial f}{\partial \varepsilon} = E_u(\varepsilon - \varepsilon^A) \tag{5a}$$

as desired, consistent with Eq. (4). The corresponding equation for σ^A is then found as

$$\sigma^A = -\frac{\partial f}{\partial \varepsilon^A} = E_u(\varepsilon - c\varepsilon^A) \tag{5b}$$

Note that the entire ADF itself may comprise several individual fields. This possibility would be useful in modeling the behavior of materials that exhibit frequency dependence weaker than that of standard anelastic solids. With the introduction of N normal ADF,[22] and using the strain-displacement relations, the material constitutive equations take the form

$$\sigma = E_u\left(u' - \sum_{i=1}^{N} u_i'^A\right) \tag{6a}$$

and

$$\sigma_i^A = E_u\left(u' - c_i u_i'^A\right) \qquad (N \text{ equations}) \tag{6b}$$

Governing Equations

The partial differential equation of motion for the total displacement field $u(x,t)$ is found in the usual way from consideration of momentum balance,

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}(\sigma) = p(x,t) \tag{7a}$$

or

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left[E_u\left(\frac{\partial u}{\partial x} - \frac{\partial u^A}{\partial x}\right)\right] = p(x,t) \tag{7b}$$

where ρ is the mass density and $p(x,t)$ is a distributed axial load. The boundary conditions require that either the total displacement $u(x)$ or the stress $\sigma(x)$ be specified at each end of the one-dimensional member; note that the stress involves both the total and anelastic strains.

The governing partial differential equation for the ADF $u^A(x,t)$ is found using a fundamental assumption of nonequilibrium thermodynamics, namely, that the rate of change of the state variable describing an irreversible process is proportional to the corresponding conjugate quantity.[23] Alternatively, the quantity σ^A may also be interpreted as a thermodynamic force driving ε^A toward an equilibrium value. This equilibrium value is that which makes σ^A zero. Then, the rate of change of ε^A is proportional to the difference between the value of ε^A and its instantaneous equilibrium value $\bar{\varepsilon}^A$,

$$\frac{\partial \varepsilon^A}{\partial t} = -\Omega(\varepsilon^A - \bar{\varepsilon}^A) = -\Omega\left(\varepsilon^A - \frac{1}{c}\varepsilon\right) = \Omega\left(\frac{1}{c}\varepsilon - \varepsilon^A\right) \tag{8a}$$

Using the strain-displacement relations, this may be expressed as

$$\frac{cE_u}{\Omega}\frac{\partial^2 u^A}{\partial x\,\partial t} = E_u\left(\frac{\partial u}{\partial x} - c\frac{\partial u^A}{\partial x}\right) \tag{8b}$$

Note that the parameter Ω corresponds to the material relaxation time at constant strain.

Following the procedure initiated in earlier work,[9,11] another form of the governing partial differential equation for the anelastic displacement field is found by taking the divergence of Eq. (8b):

$$\frac{\partial}{\partial x}\left(\frac{cE_u}{\Omega}\frac{\partial^2 u^A}{\partial x \partial t}\right) - \frac{\partial}{\partial x}\left[E_u\left(\frac{\partial u}{\partial x} - c\frac{\partial u^A}{\partial x}\right)\right] = 0 \tag{9}$$

Because the ADF cannot be directly affected through the action of external forces, but only through coupling with the total displacement field, they are effectively internal fields. Consequently, there are no geometric boundary conditions for the ADF analogous to those for the total displacement field. There are, however, force-type boundary conditions: the anelastic stress is proportional to the anelastic strain rate [compare to Eqs. (8b) and (5b)],

$$\sigma^A = (cE_u/\Omega)(\dot{u}'^A) \tag{10}$$

Note that the governing differential equations and boundary conditions for this ADF model can also be developed using an extended Hamilton's principle, a method familiar to structural dynamics engineers.[24] In that case, the strain energy, the kinetic energy, and the virtual work expressions are given as follows:

$$U(x,t) = \frac{1}{2}\int_0^L \begin{Bmatrix}u'\\ u'^A\end{Bmatrix}^T\begin{bmatrix}E_u & -E_u\\ -E_u & cE_u\end{bmatrix}\begin{Bmatrix}u'\\ u'^A\end{Bmatrix}A\,dx \tag{11a}$$

$$T(x,t) = \frac{1}{2}\int_0^L \begin{Bmatrix}\dot{u}\\ \dot{u}^A\end{Bmatrix}^T\begin{bmatrix}\rho & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}\dot{u}\\ \dot{u}^A\end{Bmatrix}A\,dx \tag{11b}$$

$$\delta W(x,t) = \int_0^L f\delta u A\,dx + \tau\delta u A|_0^L + \int_0^L\left(-\frac{cEu}{\Omega}\dot{u}'^A\right)\delta u'^A A\,dx \tag{11c}$$

For the case of a uniform member (density and modulus constant) with no distributed external loads, the governing differential equations may be summarized as

$$\rho\ddot{u} - E_u u'' + E_u u''^A = 0 \tag{12a}$$

$$(cE_u/\Omega)\dot{u}''^A - E_u u'' + cE_u u''^A = 0 \tag{12b}$$

These equations may be compared to those developed previously using a single scalar ATF, ξ:

$$\rho\ddot{u} - E_u u'' = -\delta\xi' \tag{13a}$$

$$\dot{\xi} + \Omega\xi = (\Omega\delta/\alpha)u' \tag{13b}$$

or, in terms of γ, the gradient of ξ,

$$\rho\ddot{u} - E_u u'' = -\delta\gamma \tag{14a}$$

$$\dot{\gamma} + \Omega\gamma = (\Omega\delta/\alpha)u'' \tag{14b}$$

Note that every term in the equation of evolution for the ADF, Eq. (12b), involves spatial derivatives; this equation has a nature fundamentally different from either of those developed using the

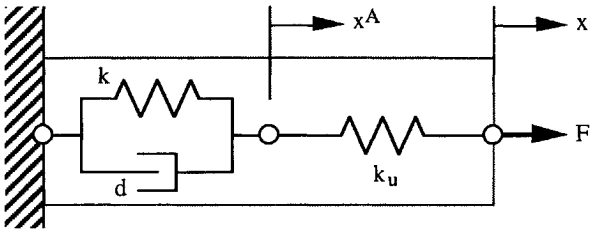

Fig. 2 Classical lumped-parameter viscoelastic model: the standard anelastic solid.

ATF formulation, Eqs. (13b) and (14b). As in that case, however, noting that u^A appears only in the form u''^A in Eqs. (12a) and (12b), that quantity (u''^A) could be interpolated directly in finite element numerical treatment of these one-dimensional equations. Because such a substitution is not possible in the three-dimensional case, however, it is not pursued here. In addition, the governing equations contain only even spatial derivatives, a result that leads to some benefits in numerical solution.

With the introduction of N normal ADF, the governing differential equations take the form

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left[E_u\left(\frac{\partial u}{\partial x} - \sum_{i=1}^{N}\frac{\partial u_i^A}{\partial x}\right)\right] = p(x,t) \quad (15a)$$

$$\frac{\partial}{\partial x}\left[\frac{E_u c_i}{\Omega_i}\frac{\partial}{\partial t}\left(\frac{\partial u_i^A}{\partial x}\right)\right] + \frac{\partial}{\partial x}\left[E_u\left(c_i\frac{\partial u_i^A}{\partial x} - \frac{\partial u}{\partial x}\right)\right] = 0$$

(N equations) (15b)

Classical Mechanical Analogy

A physical interpretation of some of the quantities involved in the present ADF model is advanced in terms of a classical mechanical analogy. Consider the lumped-parameter system shown in Fig. 2, comprising a parallel spring-dashpot unit in series with a spring unit. In this system,

x = total displacement
x^A = anelastic displacement
k_u = unrelaxed stiffness
k = stiffness
d = viscous damper
F = force on the system

Constitutive Equations

Because of the series arrangement, the force in the spring-dashpot unit and the force in the spring unit are the same. Thus,

$$F = k_u(x - x^A) \quad (16)$$

The similarity of this equation to Eq. (5a), the material constitutive equation for stress, is apparent. In addition, the force in the spring-dashpot unit is given by

$$F = kx^A + d\dot{x}^A \quad (17)$$

Rearranging, the force in the dashpot may be expressed as

$$d\dot{x}^A = k_u(x - x^A) - kx^A = k_u\{x - [1 + (k/k_u)]x^A\} \quad (18)$$

Interpreting the force in the dashpot as an anelastic force, it may now be expressed as

$$F^A = k_u(x - cx^A) \quad \text{where} \quad c = [1 + (k/k_u)] = [(k_u + k)/k_u] \quad (19)$$

The similarity of this equation to Eq. (5b), the material constitutive equation for the anelastic stress, is also apparent.

Equation of Evolution

Note that when this anelastic force F^A is zero, the system is in equilibrium (x and x^A are constant). In addition, the time rate of change of x^A is proportional to F^A:

$$\dot{x}^A(1/d)F^A = (k_u/d)(x - cx^A) \quad (20)$$

And noting that the equilibrium value of x^A (that which makes $F^A = 0$) is

$$\bar{x}^A = (1/c)x \quad \text{or} \quad x = c\bar{x}^A \quad (21)$$

the time rate of change of x^A may be expressed alternatively as

$$\dot{x}^A = -(k_u c/d)[x^A - (1/c)x] = -\Omega(x^A - \bar{x}^A) \quad (22)$$

in which Ω, the inverse of the relaxation time constant at constant displacement, is given by

$$\Omega = k_u c/d = (k_u + k)/d \quad (23)$$

The similarity of Eq. (22) to the equation of evolution for the anelastic strain, Eq. (8), is apparent.

Relaxation Strength

The relaxation strength is one useful measure of the degree of anelasticity and damping exhibited by a material.[22] It is conveniently determined here from the difference between the low- and high-frequency asymptotic moduli or stiffnesses, as follows:

$$k_{\text{high}} = k_{\text{low}}(1 + \Delta) \quad (24)$$

The low-frequency modulus is that found when the anelastic displacement takes on its equilibrium value or, equivalently, when the dashpot carries no load:

$$F = k_{\text{low}}x = k_u(x - \bar{x}^A) = k_u[x - (1/c)x] \quad (25)$$

thus,

$$k_{\text{low}} = k_u[(c - 1)/c] = k_u k/(k_u + k) \quad (26)$$

The high-frequency modulus is that found when the anelastic displacement is zero or, equivalently, when the dashpot carries the entire load:

$$F = k_{\text{high}}x = k_u(x - 0) \quad (27)$$

thus,

$$k_{\text{high}} = k_u \quad (28)$$

and the relaxation magnitude Δ is found as

$$\Delta = 1/(c - 1) = k_u/k \quad (29)$$

The elements of the mechanical system may now be relabeled to illustrate the analogy with the continuous case involving an anelastic displacement field. Figure 3 shows the system labeled in terms of the relaxation magnitude Δ, whereas Fig. 4 employs the constitutive parameter c.

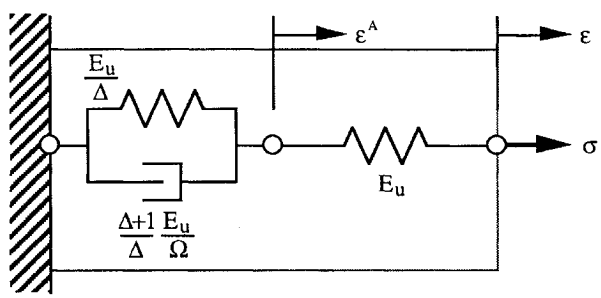

Fig. 3 Mechanical analogy for single ADF using relaxation magnitude.

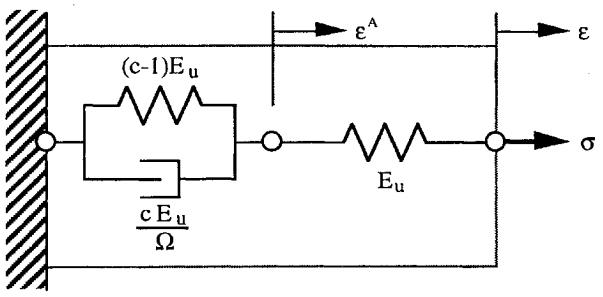

Fig. 4 Mechanical analogy for single ADF using constitutive coupling parameter.

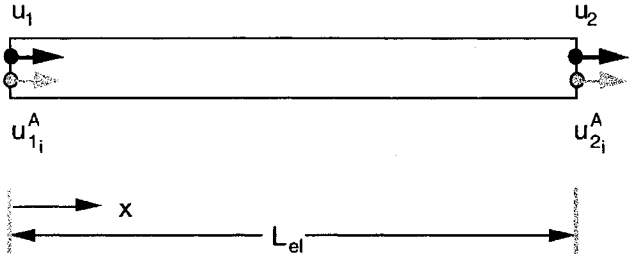

Fig. 5 One-dimensional ADF finite element.

Energy Dissipation Rate

Finally, note that the energy dissipation rate (the rate at which energy of vibration is converted to heat) is given by, in the case of the mechanical analogy,

$$H = F^A \dot{x}^A \tag{30}$$

In the one-dimensional continuum case, the energy dissipation density rate is closely related to the distributed entropy source strength and is given by

$$h = \sigma^A \dot{\varepsilon}^A \tag{31}$$

Finite Elements for a One-Dimensional Rod

This section describes the development of finite elements for one-dimensional structural members based on ADF. Because the anelastic displacement field(s) and the total displacement field may be treated similarly, a key practical benefit of this ADF approach is that it leads to more straightforward development of finite elements.

Consider a single element of length L_{el} and cross-sectional area A, as shown in Fig. 5. The total displacement field over the element, $u(x)$, is approximated using a linearly varying interpolation function, and each of the (possibly multiple) anelastic displacement fields $u_i^A(x)$ are identically approximated. The nodal degrees of freedom are the total and anelastic displacements at each end of the element.

In the absence of true variational principles governing the nonconservative behavior of interest, the method of weighted residuals may be used to develop finite element matrices from the governing partial differential equations, Eqs. (12a) and (12b).[9] For a single ADF, the resulting elemental equations may be expressed as

$$\begin{bmatrix} m/3 & m/6 \\ m/6 & m/3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1^A \\ u_2^A \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{32a}$$

$$\begin{bmatrix} ck/\Omega & -ck/\Omega \\ -ck/\Omega & ck/\Omega \end{bmatrix} \begin{Bmatrix} \dot{u}_1^A \\ \dot{u}_2^A \end{Bmatrix} - \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} ck & -ck \\ -ck & ck \end{bmatrix} \begin{Bmatrix} u_1^A \\ u_2^A \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{32b}$$

where k and m are defined as

$$m = \rho A L_{el} \qquad k = E_u A / L_{el}$$

Combined into a set of equations in second-order form, classical mass and damping and stiffness matrices are evident:

$$\left[\begin{array}{cc:cc} m/3 & m/6 & 0 & 0 \\ m/6 & m/3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \cdots \\ \ddot{u}_1^A \\ \ddot{u}_2^A \end{Bmatrix} + \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & ck/\Omega & -ck/\Omega \\ 0 & 0 & -ck/\Omega & ck/\Omega \end{array}\right] \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \cdots \\ \dot{u}_1^A \\ \dot{u}_2^A \end{Bmatrix} + \left[\begin{array}{cc:cc} k & -k & -k & k \\ -k & k & k & -k \\ \hdashline -k & k & ck & -ck \\ k & -k & -ck & ck \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_1^A \\ u_2^A \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{Bmatrix} \tag{33a}$$

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} - \{f\} \tag{33b}$$

Note the details of the structure of this matrix equation, particularly the presence of submatrices that are simple multiples of the basic elastic stiffness matrix. Once the mass and elastic stiffness matrices have been determined, the remaining submatrices may be readily determined without complicated calculations.

These equations may readily be combined in first-order state–space form:

$$\left[\begin{array}{ccc:ccc} m/3 & 0 & 0 & m/6 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & ck/\Omega & 0 & 0 & -ck/\Omega \\ \hdashline m/6 & 0 & 0 & m/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -ck/\Omega & 0 & 0 & ck/\Omega \end{array}\right] \begin{Bmatrix} \ddot{u}_1 \\ \dot{u}_1 \\ \dot{u}_1^A \\ \cdots \\ \ddot{u}_2 \\ \dot{u}_2 \\ \dot{u}_2^A \end{Bmatrix} + \left[\begin{array}{ccc:ccc} 0 & k & -k & 0 & -k & k \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & ck & 0 & k & -ck \\ \hdashline 0 & -k & k & 0 & k & -k \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & k & -ck & 0 & -k & ck \end{array}\right] \begin{Bmatrix} \dot{u}_1 \\ u_1 \\ u_1^A \\ \cdots \\ \dot{u}_2 \\ u_2 \\ u_2^A \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \\ 0 \\ \cdots \\ f_2 \\ 0 \\ 0 \end{Bmatrix} \tag{34}$$

For multiple ADF, the resulting elemental equations may be expressed in first-order form as

$$
\begin{bmatrix}
m/3 & 0 & 0 & \cdots & m/6 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & 0 & c_i k/\Omega_i & 0 & 0 & 0 & -c_i k/\Omega_i & 0 \\
\vdots & \vdots & 0 & \ddots & \vdots & \vdots & 0 & \ddots \\
m/6 & 0 & 0 & \cdots & m/3 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \\
0 & 0 & -c_i k/\Omega_i & 0 & 0 & 0 & c_i k/\Omega_i & 0 \\
\vdots & \vdots & 0 & \ddots & \vdots & \vdots & 0 & \ddots
\end{bmatrix}
\begin{Bmatrix}
\ddot{u}_1 \\ \dot{u}_1 \\ \cdots \\ \dot{u}_{1_i}^A \\ \vdots \\ \ddot{u}_2 \\ \dot{u}_2 \\ \cdots \\ \dot{u}_{2_i}^A \\ \vdots
\end{Bmatrix}
+
\begin{bmatrix}
0 & k & -k & \cdots & 0 & -k & k & \cdots \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & -k & c_i k & 0 & 0 & k & c_i k & 0 \\
\vdots & \vdots & 0 & \ddots & \vdots & \vdots & 0 & \ddots \\
0 & -k & k & \cdots & 0 & k & -k & \cdots \\
0 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots \\
0 & k & -c_i k & 0 & 0 & -k & c_i k & 0 \\
\vdots & \vdots & 0 & \ddots & \vdots & \vdots & 0 & \ddots
\end{bmatrix}
\begin{Bmatrix}
\dot{u}_1 \\ u_1 \\ \cdots \\ u_{1_i}^A \\ \vdots \\ \dot{u}_2 \\ u_2 \\ \cdots \\ u_{2_i}^A \\ \vdots
\end{Bmatrix}
=
\begin{Bmatrix}
f_1 \\ 0 \\ \cdots \\ 0 \\ \vdots \\ f_2 \\ 0 \\ \cdots \\ 0 \\ \vdots
\end{Bmatrix}
\tag{35}
$$

Modal Analysis

To illustrate the performance of this ADF-damped one-dimensional element, a specific boundary-value eigenvalue problem was addressed, namely, the determination of the natural modes of vibration of a fixed-free member. Such analyses have been conducted by one of the authors using the matrix programming capabilities of a commercial finite element code (MSC/NASTRAN®). For developmental purposes, between 10 and 50 finite elements were typically used to model the structural member. Assembly of global augmented mass and stiffness matrices followed a procedure identical to that employed in conventional finite element analysis of elastic structures.

For this example, a stiff polymerlike material exhibiting a peak loss factor η of 0.2 at a frequency of 1000 rad/s is considered. Table 1 summarizes the material and structural parameter values used in this single-ADF example.

For materials exhibiting frequency dependence weaker than that of a standard anelastic solid, multiple-ADF finite elements may be used. Single- or multiple-ADF model parameters may be determined from measured material frequency-dependent complex modulus data using a curve fitting approach.[11] Increasing the number of ADFs increases the accuracy with which material behavior is described.

Figure 6 shows the results of a one-dimensional modal analysis for a member made from this material. The analysis used 15 elements. Over the frequency range of interest, the structural modal damping ratios increase to a peak value, then decrease, as expected. Note that the numerical values of the modal frequencies calculated from the finite element analysis only approximate the actual solutions to the partial differential equations, with accuracy generally decreasing with increasing mode number and increasing with the number of elements. However, this model evidently preserves the relative magnitudes of the imaginary and real parts of complex eigenvalues, even when the absolute magnitude is not well approximated.

The significance of the good agreement between analytical and computational results for this simple problem is that the accuracy of the computational tools may be expected to carry over to applications involving complex, irregular structures made from different materials.

Table 1 Parameter values used in single-ADF example

Fundamental quantities		
Relaxed (static) modulus	E_R	= 7.0 GPa
Peak loss factor	η_p	= 0.20
Frequency of peak damping	ω_p	= 1000 rad/s
Structural member length	L	= 25.0 m
Material density	ρ	= 1500 kg/m^3
Derived quantities		
Relaxation magnitude	Δ	= $2\eta_p[\eta_p + \sqrt{(1+\eta_p^2)}] = 0.4879$
Constitutive coupling parameter	c	= $(1+\Delta)/\Delta = 3.0495$
Unrelaxed modulus	E_U	= $E_R(1+\Delta) = 10.4155$ GPa
Inverse of relaxation time at constant strain	Ω	= $\omega_p(1+\Delta)^{1/2} = 1219.8$ rad/s

Frequency-Response Analysis

For additional illustration of the performance of this one-dimensional ADF-based element, frequency response analyses were performed using the model described in the preceding paragraphs. Such analyses have also been conducted using the general matrix capabilities of a commercial finite element code. The frequency response function of interest in this example is that between a force applied at the free tip of the structure and the corresponding displacement at the same point.

Figure 7 shows frequency response functions for two structural models, a baseline model and the ADF model. The baseline model is a conventional elastic model with constant modal damping selected to match that of the fundamental mode of the ADF model. This is

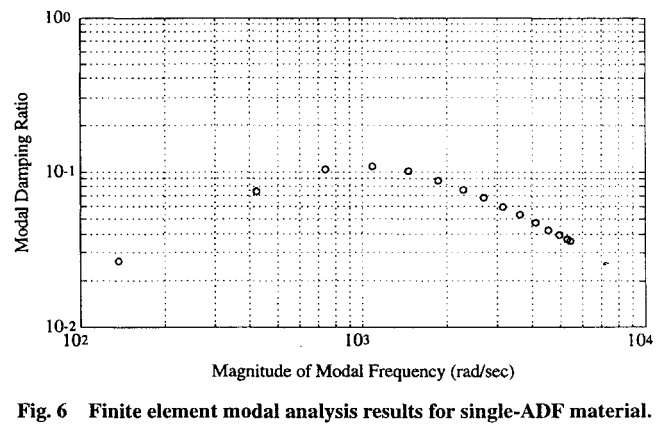

Fig. 6 Finite element modal analysis results for single-ADF material.

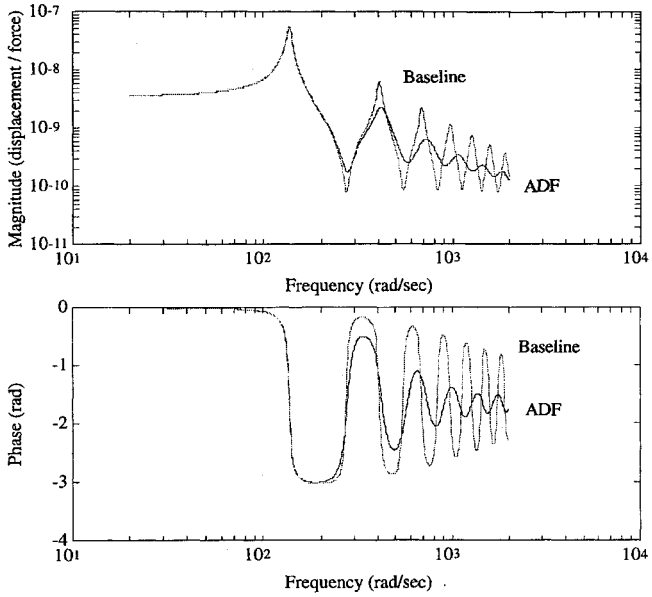

Fig. 7 Frequency response functions for baseline and ADF models.

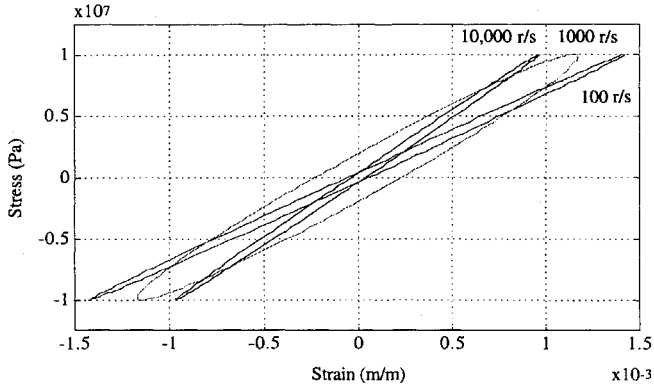

Fig. 8 Stress–strain hysteresis loops for ADF model at three frequencies.

motivated by the classical notion of hysteretic damping, or approximate frequency independence of damping.

Note that the ADF modeling method accurately captures the frequency dependence of material damping. In addition, it captures the frequency dependence of material stiffness through its effect on modal frequencies. This stiffness variation is indicated by increasing modal frequencies from the baseline to the ADF model, with increasing frequency. The two models have the same static elastic behavior.

Stress–Strain Hysteresis

Stress–strain hysteresis loops are alternatively used to describe material damping behavior without reference to resonant structural vibration. For illustration, Fig. 8 shows the force-displacement behavior of the material postulated in the preceding discussion at several frequencies: below, at, and above the frequency at which peak damping is observed. The ADF model evidently preserves the frequency dependence of damping, as indicated by the different areas and aspect ratios of the ellipses. In addition, it captures the frequency dependence of material stiffness, as indicated both by the different slopes of the major axes of the ellipses, as well as the varying peak displacement amplitudes.

Conclusion

A physically motivated approach to modeling the dynamic behavior of viscoelastic materials and structures based on the use of ADF is reported. This approach has been illustrated through the development of the material constitutive equations, as well as the governing differential equations and boundary conditions for a one-dimensional structural member, and a physical interpretation of some of the quantities involved has been advanced in terms of a classical mechanical analogy. Because the ADF and the total displacement field may be treated similarly in analytical or numerical study, an important practical benefit of this ADF approach is that it leads to more straightforward development of finite elements. Example modal analyses and frequency-response analyses of a one-dimensional structural member were performed using the general matrix manipulation capabilities of a commercial finite element code. These analyses, along with a stress–strain hysteresis study, demonstrated the ability of the model to capture the characteristic frequency dependence of viscoelastic material stiffness and damping.

Finally, the concept of an ADF provides a basis for the development of a general three-dimensional modeling approach. Such an approach is required for describing viscoelastic material behavior under complex states of stress. Anticipated aerospace applications of the ADF modeling approach include damping treatments, elastomeric bearings and dampers, solid propellant rocket motors, nondestructive evaluation of composite structures, and statistical energy analysis of structural acoustics.

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